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The heat content asymptotics for variable geometries

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Abstract. Let g_t be a time-dependent family of Riemannian metrics on a manifold M with a smooth boundary. Let ϕ be the initial temperature of M and let ρ be the specific heat of M. Impose Dirichlet or Neumann boundary conditions and let $\beta(t)$ be the resulting total heat energy content of M. As $t \downarrow 0$, one can expand $\beta \sim \sum_n \beta_n t^{n/2}$ in an asymptotic series in half integer powers of the parameter t. We determine β_n for $n \leqslant 4$ in terms of geometric quantities; this extends previous results from the autonomous setting where the metric was independent of the parameter t to a dynamic setting where the metric is permitted to be time dependent.

Let M be a compact manifold with smooth boundary ∂M. Let g_t be a smooth one-parameter family of metrics on M and let Δ_t be the associated Laplace operators defined by these metrics. If ϕ is the initial temperature of M, the temperature distribution $u(x;t)$ for $t \geqslant 0$ is determined by the equations:

$$(\partial_t + \Delta_t)u = 0 \qquad \mathcal{B}u = 0 \qquad \text{and} \qquad u(x;0) = \phi.$$

Here $\mathcal{B}$ denotes either Dirichlet or Neumann boundary conditions; we can impose different boundary conditions on different components of the boundary if we wish. Let $\rho(x;t)$ be the specific heat of the manifold. The total heat energy content of the manifold is given by

$$\beta(\phi,\rho)(t) := \int_M u(x;t)\rho(x;t)\,\mathrm{d}x$$

where $\mathrm{d}x$ is the Riemannian measure determined by the metric at time $t = 0$; this does not involve any loss of generality since we permit ρ to vary with time. As $t \downarrow 0$, there is an asymptotic series of the form

$$\beta(\phi,\rho)(t) \sim \sum_{n\geqslant 0} \beta_n(\phi,\rho)t^{n/2}.$$

The focus of this paper will be the calculation of the invariants β_n in this setting for $n = 0, 1, 2, 3, 4$; we must extend previous results from the autonomous setting to the time-dependent setting. We will use methods of invariance theory to do this. Consequently, although we are primarily interested in the scalar Laplacian, it is convenient to enlarge the class of operators with which we work; this enlarges the number of functorial properties at our disposal. It is somewhat paradoxical in this subject that applying the functorial method requires one to determine formulae in great generality even if one is just, interested in the scalar Laplacian. We refer, for example to step 3 in the proof of lemma 9 where we use a non-trivial connection.

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Any operator of Laplace type on a vector bundle V can be put in the form $D_0 = -(\mathrm{Trace}(\nabla^2) + E)$ where ∇ is a connection on V and E is an endomorphism of V; we refer to [2] for details. (Since we must deal with non-trivial connections and endomorphisms, we are naturally led to the bundle case.) We consider time-dependent families of operators of Laplace type which have the form

$$D_t := D_0 + \sum_{r>0} t^r \{\mathcal{G}_{r,ij}(x)\nabla_i\nabla_j + \mathcal{F}_{r,i}(x)\nabla_i + \mathcal{E}_r(x)\}. \tag{1}$$

(We expand relative to a frame for the tangent bundle which is orthogonal with respect to the original metric g_0.) The metric has the form

$$\mathrm{d}s^2 = \left(g_{ij} + \sum_{r>0} t^r \mathcal{G}_{r,ij}(x)\right) \mathrm{d}x^i \circ \mathrm{d}x^j.$$

The first-order perturbation of the operator is given by $\mathcal{F}$, and the second-order perturbation is given by $\mathcal{E}$. The scalar Laplacian Δ_t defined by the metric $\mathrm{d}s^2(t)$ can be put in this form. The main result of this paper is contained in theorem 4 which expresses the asymptotic coefficients β_n for $n \leqslant 4$ in the heat content expansion in terms of geometric quantities.

Previous results in this area dealt with time-independent (i.e. autonomous) processes; we shall summarize the results of computations performed in [1, 3, 9] in theorem 3 as these computations form the base for the extension to time-dependent processes described in theorem 4. There are many other results in this area and many different techniques which have been employed. For example, van den Berg and Le Gall [6] use probablistic methods to study these coefficients. van den Berg and Srisatkunarajah [7] study polygonal regions in the plane; we shall restrict ourselves to the smooth setting. McAvity [12, 13] uses a modified de Witt ansatz to study these coefficients. Savo [14, 15] uses techniques from functional analysis to derive a recursive formula for all the coefficients if the operator in question is the time-independent Laplacian and if $\rho = \phi = 1$. See also [8] for related work.

We introduce the following notational conventions. Denote the Riemannian measures on M and on ∂M by $\mathrm{d}x$ and $\mathrm{d}y$. Let D_0 be an operator of Laplace type on the space of smooth sections $C^\infty(V)$ to a vector bundle over M. Recall that there exists a unique connection ∇ on V and a unique endomorphism E of V so that

$$D_0 = -\{\mathrm{Trace}\ \nabla^2 + E\}.$$

If $D_0 = \Delta_0$ is the scalar Laplacian, then the connection ∇ is flat and $E = 0$. More generally, if $D = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B)$ is an operator of Laplace type, then (see [2, 10]) the connection one-form ω of ∇ and the endomorphism E are given by

$$\omega_\delta = \tfrac{1}{2} g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu) \qquad \text{and} \qquad E = b - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma). \tag{2}$$

We assume a decomposition of the boundary ∂M as the disjoint union of two closed sets C_D and C_N. We consider the boundary operator

$$\mathcal{B}u := u|_{C_D} \oplus (u_{;m} + Su)|_{C_N}.$$

Here $u_{;m}$ is the inward unit normal covariant derivative of u and S is an auxiliary endomorphism of V. This formalism permits us to treat both the Robin and Dirichlet boundary conditions. Let ϕ give the initial temperature distribution of the manifold and let $u(x;t) = u_{\phi,D_0}(x;t)$ be the temperature distribution for $t > 0$;

$$(\partial_t + D_0)u = 0 \qquad \mathcal{B}u = 0 \qquad \text{and} \qquad u|_{t=0} = \phi.$$

Let ρ be a smooth section to the dual bundle V^* giving the specific heat of the manifold. Then the total energy content of the manifold is given by:

$$\beta(\phi, D_0, \rho)(t) := \int_M \langle u(x;t), \rho(x)\rangle\, \mathrm{d}x.$$

In this expression, $\langle\cdot,\cdot\rangle$ denotes the natural pairing between V and V^*. As $t \downarrow 0$ there is an asymptotic series of the form

$$\beta(\phi, D_0, \rho)(t) \sim \sum_{n\geqslant 0} t^{n/2}\beta_n(\phi, D_0, \rho).$$

There exist invariants which are locally computable so that

$$\beta_n(\phi, D_0, \rho) = \int_M \beta_n^M(\phi, D_0, \rho)\, \mathrm{d}x + \int_{\partial M} \beta_n^{\partial M}(\phi, D_0, \rho)\, \mathrm{d}y.$$

These are the heat content asymptotics which describe the short-time heat flow defined by the problem. These results follow easily using methods developed in [11].

Let $\tilde{D}_0$, $\tilde{\nabla}$, and $\tilde{\mathcal{B}}$ be the adjoint operators on the dual bundle V^*. Let indices i and j range from 1 to m and index a local orthonormal frame field $\{e_i\}$ for the tangent bundle of M; on the boundary, we normalize the frame so that e_m is the inward unit normal and let indices a, b, and c range from 1 to $m-1$ and index the induced orthonormal frame for the tangent bundle of the boundary. We adopt the Einstein convention and sum over repeated indices. Let ';' and ':' denote multiple covariant differentiation with respect to the Levi-Civita connections of M and of ∂M. Let R be the Riemann curvature tensor, let $\mathcal{R}$ be the scalar curvature, and let L be the second fundamental form defined by the metric g_0. Let Ω be the curvature of the connection ∇. The following result follows from computations performed in [1, 3, 9].

Theorem 3. *With the notation established above, we have:*

(1) $\beta_0(\phi, D_0, \rho) = \int_M \langle\phi, \rho\rangle\, \mathrm{d}x$.
(2) $\beta_1(\phi, D_0, \rho) = -2\pi^{-1/2}\int_{C_D}\langle\phi, \rho\rangle\, \mathrm{d}y$.
(3) $\beta_2(\phi, D_0, \rho) = -\int_M\langle D_0\phi, \rho\rangle\, \mathrm{d}x + \int_{C_D}\{\langle\frac{1}{2}L_{aa}\phi, \rho\rangle - \langle\phi, \rho_{;m}\rangle\}\, \mathrm{d}y + \int_{C_N}\langle\mathcal{B}\phi, \rho\rangle\, \mathrm{d}y$.
(4) $\beta_3(\phi, D_0, \rho) = -2\pi^{-1/2}\int_{C_D}\{\frac{2}{3}\langle\phi_{;mm}, \rho\rangle + \frac{2}{3}\langle\phi, \rho_{;mm}\rangle - \langle\phi_{:a}, \rho_{:a}\rangle + \langle E\phi, \rho\rangle - \frac{2}{3}L_{aa}\langle\phi_{;m}, \rho\rangle - \frac{2}{3}L_{aa}\langle\phi, \rho_{;m}\rangle + \langle(\frac{1}{12}L_{aa}L_{bb} - \frac{1}{6}L_{ab}L_{ab} + \frac{1}{6}R_{amam})\phi, \rho\rangle\}\, \mathrm{d}y + \frac{2}{3}\cdot 2\pi^{-1/2}\int_{C_N}\langle\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle\, \mathrm{d}y$,
(5) $\beta_4(\phi, D_0, \rho) = \frac{1}{2}\int_M\langle D_0\phi, \tilde{D}_0\rho\rangle + \int_{C_D}\{\frac{1}{2}\langle(D_0\phi)_{;m}, \rho\rangle + \frac{1}{2}\langle\phi, (\tilde{D}_0\rho)_{;m}\rangle - \frac{1}{4}\langle L_{aa}D_0\phi, \rho\rangle - \frac{1}{4}\langle L_{aa}\phi, \tilde{D}_0\rho\rangle + \langle(\frac{1}{8}E_{;m} - \frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac{1}{8}L_{ab}L_{ac}L_{bc} - \frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac} + \frac{1}{32}\mathcal{R}_{;m} + \frac{1}{16}L_{ab:ab})\phi, \rho\rangle - \frac{1}{4}L_{ab}\langle\phi_{:a}, \rho_{:b}\rangle - \frac{1}{8}\langle\Omega_{am}\phi_{:a}, \rho\rangle + \frac{1}{8}\langle\Omega_{am}\phi, \rho_{:a}\rangle\}\, \mathrm{d}y + \int_{C_N}\{-\frac{1}{2}\langle\mathcal{B}\phi, \tilde{D}_0\rho\rangle - \frac{1}{2}\langle D_0\phi, \tilde{\mathcal{B}}f_2\rangle + \langle(\frac{1}{2}S + \frac{1}{4}L_{aa})\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle\, \mathrm{d}y$.

Although the assumption that the underlying geometry is autonomous is natural in many situations, there are some physical situations in which the geometry is time dependent; the Universe evolves with time for example.

Let D_t be the time-dependent family of operators of Laplace type given in equation (1). Let $u = u_{\phi,D}$ be the temperature distribution defined by the equations:

$$(\partial_t + D_t)u = 0 \qquad \mathcal{B}u = 0 \qquad \text{and} \qquad u|_{t=0} = \phi.$$

Let β and β_n be the associated heat content function and heat content asymptotics:

$$\beta(\phi, D, \rho)(t) := \int_M u(x;t)\rho(x;t)\, \mathrm{d}x \sim \sum_{n\geqslant 0}\beta_n(\phi, D, \rho)t^{n/2}.$$

If the specific heat ρ is time dependent, expand $\rho(x;t) = \sum_{0\leqslant k<k_0} \rho_k(x)t^k + \mathrm{O}(t^{k_0})$ in a Taylor series. Then it is immediate that

$$\beta_n(\phi, D, \rho) = \sum_{2k+\ell=n} \beta_l(\phi, D, \rho_k).$$

Consequently, we assume that $\rho = \rho(x)$ henceforth is autonomous. The following is the main theorem of this paper. It gives the new terms which appear in the asymptotic expansion when Laplacian is time dependent.

Theorem 4. *(1)* $\beta_0(\phi, D, \rho) = \beta_0(\phi, D_0, \rho)$.
(2) $\beta_1(\phi, D, \rho) = \beta_1(\phi, D_0, \rho)$.
(3) $\beta_2(\phi, D, \rho) = \beta_2(\phi, D_0, \rho)$.
(4) $\beta_3(\phi, D, \rho) = \beta_3(\phi, D_0, \rho) + \frac{1}{2\sqrt{\pi}} \int_{C_D} \mathcal{G}_{1,mm}\phi\rho \,\mathrm{d}y$.
(5) $\beta_4(\phi, D, \rho) = \beta_4(\phi, D_0, \rho) - \frac{1}{2}\int_M \{\mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi\}\rho \,\mathrm{d}x + \int_{C_D}\{(\frac{7}{16}\mathcal{G}_{1,mm;m} - \frac{1}{4}\mathcal{G}_{1,mm}L_{aa} - \frac{5}{16}\mathcal{F}_{1,m})\phi\rho \frac{5}{16}\mathcal{G}_{1,am}\phi_{:a}\rho + \frac{1}{2}\mathcal{G}_{1,mm}\rho_{;m}\phi\} \,\mathrm{d}y - \frac{1}{2}\int_{C_N} \mathcal{G}_{1,mm}\rho\mathcal{B}\phi \,\mathrm{d}y$.

Remark. *We now return to the original problem. Let* $G_{ij}(t) := g_{ij} + th_{ij} + \cdots$ *be a time dependent metric; we use theorem 4 to see that the terms which are* $\mathrm{O}(t^2)$ *do not play a role in* β_n *for* $n \leqslant 4$. *Let* $G := \det(G_{ij})$. *We then have*

$$\Delta_t = -G^{-1}\partial_i G^{ij} G \partial_j.$$

We choose local coordinates so $g_{ij}(x_0) = \delta_{ij}$ *and so* $\partial_k g_{ij}(x_0) = 0$. *We may then expand the Laplacian at the point* x_0 *in the form:*

$$\begin{aligned}\Delta_t &= -(1 - th_{kk})\partial_i(\delta_{ij} - th_{ij})(1 + th_{\ell\ell})\partial_j \\ &= \Delta_0 + th_{ij}\partial_i\partial_j + t(h_{ij;i} - h_{\ell\ell;j})\partial_j + \mathrm{O}(t^2).\end{aligned}$$

Consequently, to apply theorem 4 to the scalar Laplacian for a time-dependent metric, we must set

$$\mathcal{G}_{1,ij} = h_{ij}, \mathcal{F}_{1,i} = h_{ij;j} - h_{jj;i} \qquad \text{and} \qquad \mathcal{E}_1 = 0.$$

Remark. *In this paper, we deal with homogeneous boundary conditions and a zero heat source. However, the methods developed in [4, 5] can easily be adapted to study variable geometry with inhomogeneous boundary conditions and a non-trivial heat source.*

We devote the remainder of this paper to the proof of theorem 4. We begin with a technical lemma involving products. We say that the structures split if $M = M_1 \times M_2$ where M_1 is closed, if $\phi = \phi_1\phi_2$, if $\rho = \rho_1\rho_2$, and if $D_t = D_{1,t} + D_{2,t}$. Then $u = u_1u_2$ so $\beta(\phi, D, \rho)(t) = \beta(\phi_1, D_1, \rho_1)(t) \cdot \beta(\phi_2, D_2, \rho_2)(t)$. This shows the following lemma.

Lemma 5. *If the structures split, then*

$$\beta_n(\phi, D, \rho) = \sum_{k+\ell=n} \beta_k(\phi_1, D_1, \rho_1)\beta_\ell(\phi_2, D_2, \rho_2).$$

There exist local invariants β_n^M and $\beta_n^{\partial M}$ which are bilinear in the covariant derivatives of the functions ϕ and ρ with coefficients which are invariant expressions in the covariant derivatives of the tensors L, S, R, Ω, E, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ so that:

$$\beta_n(\phi, D, \rho) = \int_M \beta_n^M(\phi, D, \rho) \,\mathrm{d}x + \int_{\partial M} \beta_n^{\partial M}(\phi, D, \rho) \,\mathrm{d}y.$$

We assign weight zero to ϕ and ρ; we assign weight one to L and S; we assign weight two to R, Ω, and E; we assign weight $2k$ to $\mathcal{G}_k$; we assign weight $2k+1$ to $\mathcal{F}_k$; we assign weight $2k+2$

to $\mathcal{E}_k$. We increase the weight by one for every explicit covariant derivative. We established the following result in the autonomous case using dimensional analysis in [2] (see lemmas 2.2 and 2.3); the same argument extends immediately to the time-dependent case so we omit details.

Lemma 6. *The local invariants β_n^M are homogeneous of weight n and the local invariants $\beta_n^{\partial M}$ are homogeneous of weight $n-1$.*

We use lemma 6 to determine the general form of the invariants β_n for $n \leqslant 4$.

Lemma 7. *Let $\rho = \rho(x)$. Then there exist universal constants so that*

(1) $\beta_0(\phi, D, \rho) = \beta_0(\phi, D_0, \rho)$.
(2) $\beta_1(\phi, D, \rho) = \beta_1(\phi, D_0, \rho)$.
(3) $\beta_2(\phi, D, \rho) = \beta_2(\phi, D_0, \rho) + \int_M \{a_1 \mathcal{G}_{1,ii}\phi\rho\}$.
(4) $\beta_3(\phi, D, \rho) = \beta_3(\phi, D_0, \rho) + \int_{C_N}\{(a_2^N \mathcal{G}_{1,aa} + c_1^N \mathcal{G}_{1,mm})\phi\rho\}\,\mathrm{d}y + \int_{C_D}\{(a_2^D \mathcal{G}_{1,aa} + c_1^D \mathcal{G}_{1,mm})\phi\rho\}\,\mathrm{d}y$.
(5) $\beta_4(\phi, D, \rho) = \beta_4(\phi, D_0, \rho) + \int_M \{(a_3 \mathcal{G}_{1,ii;jj} + a_4 \mathcal{G}_{1,ij;ij} + a_5 \mathcal{G}_{1,ii}\mathcal{G}_{1,jj} + a_6 \mathcal{G}_{1,ij}\mathcal{G}_{1,ij} + a_7 \mathcal{G}_{2,ii})\phi + a_8 \mathcal{G}_{1,ii}\phi_{;jj} + a_9 \mathcal{F}_{1,i;i}\phi + a_{13}\mathcal{G}_{1,ii}E\phi + (b_1 \mathcal{G}_{1,jj;i} + b_2 \mathcal{G}_{1,ij;j})\phi_{;i} + c_2 \mathcal{G}_{1,ij}\phi_{;ij} + c_6 \mathcal{E}_1\phi + d_2 \mathcal{F}_{1,i}\phi_{;i}\}\rho\,\mathrm{d}x + \int_{C_N}\{a_{10}^N \mathcal{G}_{1,aa}\phi_{;m}\rho + a_{11}^N \mathcal{G}_{1,aa}\phi\rho_{;m} + a_{12}^N \mathcal{G}_{1,aa}L_{bb}\phi\rho + (a_{14}^N \mathcal{G}_{1,aa}S + c_7^N \mathcal{G}_{1,mm}S)\phi\rho + (b_3^N \mathcal{G}_{1,aa;m} + b_4^N \mathcal{G}_{1,am:a} + b_5^N \mathcal{G}_{1,ab}L_{ab})\phi\rho + c_3^N \mathcal{G}_{1,mm}\phi_{;m}\rho + c_4^N \mathcal{G}_{1,mm}\phi\rho_{;m} + c_5^N \mathcal{G}_{1,mm}L_{aa}\phi\rho + d_3^N \mathcal{G}_{1,mm;m}\rho\phi + d_4^N \mathcal{G}_{1,am}\phi_{:a}\rho + d_5^N \mathcal{F}_{1,m}\phi\rho\}\,\mathrm{d}y + \int_{C_D}\{a_{10}^D \mathcal{G}_{1,aa}\phi_{;m}\rho + a_{11}^D \mathcal{G}_{1,aa}\phi\rho_{;m} + a_{12}^D \mathcal{G}_{1,aa}L_{bb}\phi\rho + (b_3^D \mathcal{G}_{1,aa;m} + b_4^D \mathcal{G}_{1,am;a} + b_5^D \mathcal{G}_{1,ab}L_{ab})\phi\rho + c_3^D \mathcal{G}_{1,mm}\phi_{;m}\rho + c_4^D \mathcal{G}_{1,mm}\phi\rho_{;m} + c_5^D \mathcal{G}_{1,mm}L_{aa}\phi\rho + d_3^D \mathcal{G}_{1,mm;m}\phi\rho + d_4^D \mathcal{G}_{1,am}\phi_{:a}\rho + d_5^D \mathcal{F}_{1,m}\phi\rho\}\,\mathrm{d}y$.

Proof. We use Weyl's theorem [16] on the invariants of the orthogonal group to express these invariants in terms of contractions of indices. We integrate by parts to exchange derivatives at the cost of introducing additional boundary terms to normalize the interior integrands so no covariant derivatives of ρ are present. Similarly, we integrate by parts on the boundary to normalize the boundary integrands so no tangential covariant derivatives of ρ are present. We write down a suitable spanning set and apply lemma 6 to see that the β_n for $n \leqslant 4$ have the form given in lemma 7 where the constants *a priori* depend on the dimension of the manifold.

Let $(M_2, D_2, \phi_2, \rho_2)$ be given. Let M_1 be the circle S^1 with the usual periodic parameter y. Let $D_1 = -\partial_y^2$ and $\phi_1 = 1$. Then $u_1 = 1$ and thus we have that $\beta(\phi_1, D_1, \rho_1) = \int_{M_1} \rho_1\,\mathrm{d}y_1$. We use lemma 5 to see

$$\beta_n(\phi, D, \rho) = \int_{M_1} \rho_1\,\mathrm{d}x_1 \cdot \beta_n(\phi_2, D_2, \rho_2).$$

It now follows that the coefficients which appear in lemma 7 are independent of the dimension and are universal constants. □

The lack of commutativity in the vector valued case does not play a role in these expressions; we therefore restrict ourselves henceforth to the scalar setting. To simplify the notation, let $\tilde{\beta}_n(\phi, D, \rho) := \beta_n(\phi, D, \rho) - \beta_n(\phi, D_0, \rho)$. We begin the proof of theorem 4 by determining the constants a_i, b_j, and c_k which appear in lemma 7.

Lemma 8.

(1) $\tilde{\beta}_3(\phi, D, \rho) = \frac{1}{2\sqrt{\pi}} \int_{C_D} \mathcal{G}_{1,mm}\phi\rho\,\mathrm{d}y$.
(2) $\tilde{\beta}_4(\phi, D, \rho) = -\frac{1}{2}\int_M \{\mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{E}_1\phi\}\rho\,\mathrm{d}x + \int_{C_D}\{-\frac{1}{4}\mathcal{G}_{1,mm}L_{aa}\phi\rho + \frac{1}{2}\mathcal{G}_{1,mm}\rho_{;m}\phi\}\,\mathrm{d}y - \frac{1}{2}\int_{C_N} \mathcal{G}_{1,mm}\rho\mathcal{B}\phi\,\mathrm{d}y + \int_M d_2 \mathcal{F}_{1,i}\phi_{;i}\rho\,\mathrm{d}x + \int_{C_D}\{d_3^D \mathcal{G}_{1,mm;m}\phi\rho + d_4^D \mathcal{G}_{1,am}\phi_{:a}\rho + d_5^D \mathcal{F}_{1,m}\phi\rho\}\,\mathrm{d}y + \int_{C_N}\{d_3^N \mathcal{G}_{1,mm;m}\rho\phi + d_4^N \mathcal{G}_{1,am}\phi_{:a}\rho + d_5^N \mathcal{F}_{1,m}\phi\rho\}\,\mathrm{d}y$.

Proof.

Step 1. We apply lemma 5. Let $(M_2, \phi_2, D_2, \rho_2)$ be arbitrary. Let

$$M_1 = \mathbb{T}^k := S^1 \times \cdots \times S^1 \qquad \text{let} \quad \phi_1 = 1$$

and let

$$D_{1,t} = \Delta_{M_1} + \sum_{r>0} t^r \Bigg(\sum_{i,j \leqslant k} \mathcal{G}_{r,ij} \nabla_i \nabla_j + \sum_{i \leqslant k} \mathcal{F}_{r,i} \nabla_i \Bigg).$$

Since $D_{1,t}\phi_1 = 0$, $u_1 = 1$ and $\beta_n(1, D_1, \rho_1) = 0$ for $n > 0$. Thus

$$\beta_n(\phi_2, D_1 + D_2, \rho_1\rho_2) = \beta_0(1, D_1, \rho_1)\beta_n(\phi_2, D_2, \rho_2).$$

In particular β_n is independent of the tensors $\mathcal{F}_{1,i}$ and $\mathcal{G}_{1,ij}$ for $i, j \leqslant k$. This shows that the following relations hold:

$$\begin{aligned} 0 &= a_1 = a_2^N = a_2^D = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 \\ &= a_9 = a_{10}^N = a_{10}^D = a_{11}^N = a_{11}^D = a_{12}^N = a_{12}^D = a_{13} = a_{14}^N. \end{aligned}$$

Consequently the higher-order Taylor coefficients $\mathcal{E}_r$, $\mathcal{F}_{r,i}$, and $\mathcal{G}_{r,ij}$ do not play a role in the computation of β_n if $n \leqslant 4$ and if $r \geqslant 2$. We may therefore restrict ourselves to first-order deformations of the Laplacian Δ henceforth and set

$$\mathcal{E} = \mathcal{E}_1 \qquad \mathcal{F}_i = \mathcal{F}_{1,i} \qquad \mathcal{G}_{ij} = \mathcal{G}_{1,ij}.$$

Step 2. Let $M := \mathbb{T}^k \times [0, 1]$, let y_a be the periodic parameters on the torus for $1 \leqslant a \leqslant k$, and let $z \in [0, 1]$ be the normal parameter. Let $f_{ab}(z)$ be functions which are close in the C^∞ topology to the Kronecker symbol δ_{ab} and let

$$\mathrm{d}s^2 := f_{ab}(z)\,\mathrm{d}y^a \circ \mathrm{d}y^b + \mathrm{d}z^2$$

define the Laplacian Δ_0. Let $\phi = \phi(z)$ and let $u_0 = u_{\phi,\Delta_0}$ be defined by the trivial variation; u_0 only depends on the normal parameter z. We take a variation where $\mathcal{E} = 0$, $\mathcal{F}_m = 0$, and $\mathcal{G}_{mm} = 0$. Therefore:

$$(\mathcal{G}_{ij}\nabla_i\nabla_j + \mathcal{F}_i\nabla_i + \mathcal{E})u_0 = 0$$

so $u_{\phi,D} = u_0$. Thus β is independent of the remaining $\mathcal{F}$ and $\mathcal{G}$ variables and

$$0 = b_1 = b_2 = b_3^N = b_3^D = b_4^N = b_4^D = b_5^N = b_5^D.$$

Step 3. Let $s = s(t) := \mathrm{e}^t - 1$; $\partial_s = \mathrm{e}^{-t}\partial_t$ and $s(0) = 0$. Consider the conformal deformation $D_t = \mathrm{e}^t D_0$. Let $u_0 := u_{\phi,D_0}$ and let $u(x; t) = u_0(x; s(t))$. Then:

$$(\partial_t + D_t)u = \mathrm{e}^t(\partial_s + D_0)u_0 = 0 \qquad \mathcal{B}u = 0 \qquad \text{and} \qquad u|_{t=0} = u_0|_{s=0} = \phi.$$

Consequently, $u_{\phi,D}(x; t) = u_{\phi,D_0}(x; s(t))$ and $\beta(\phi, D, \rho)(t) = \beta(\phi, D_0, \rho)(s(t))$. We have

$$\begin{aligned} &s^{\frac{1}{2}} = t^{\frac{1}{2}} + \tfrac{1}{4}t^{\frac{3}{2}} + \mathrm{O}(t^{\frac{5}{2}}) \qquad s = t + \tfrac{1}{2}t^2 + \mathrm{O}(t^3) \\ &s^{\frac{3}{2}} = t^{\frac{3}{2}} + \mathrm{O}(t^{\frac{5}{2}}) \qquad s^2 = t^2 + \mathrm{O}(t^3). \end{aligned}$$

We equate coefficients of t in the asymptotic expansions

$$\sum_n t^n \beta_n(\phi, D, \rho) \sim \beta(\phi, D, \rho)(t) = \beta(\phi, D_0, \rho)(s(t)) \sim \sum_n \beta_n(\phi, D_0, \rho)s(t)^n$$

and use theorem 3 to derive the relationships:

$$\tilde\beta_3(\phi, D, \rho) = \tfrac{1}{4}\beta_1(\phi, D_0, \rho) = -\frac{1}{2\sqrt{\pi}}\int_{C_D}\phi\rho\,\mathrm{d}y$$

$$\tilde\beta_4(\phi, D, \rho) = \tfrac{1}{2}\beta_2(\phi, D_0, \rho) = \tfrac{1}{2}\int_M(\phi_{;ii} + E\phi)\rho\,\mathrm{d}x + \tfrac{1}{2}\int_{C_N}\rho(\phi_{;m} + S\phi)\,\mathrm{d}y$$
$$+\tfrac{1}{2}\int_{C_D}\{\tfrac{1}{2}L_{aa}\phi\rho - \rho_{;m}\phi\}\,\mathrm{d}y.$$

In this setting, we have $\mathcal{E} = -E$, $\mathcal{F} = 0$, and $\mathcal{G} = -g_0$. We may therefore complete the proof of the lemma by deriving the relationships:

$$c_1^N = 0 \qquad c_1^D = \frac{1}{2\sqrt{\pi}} \qquad c_2 = -\tfrac{1}{2} \qquad c_3^N = -\tfrac{1}{2} \qquad c_3^D = 0$$
$$c_4^N = 0 \qquad c_4^D = \tfrac{1}{2} \qquad c_5^N = 0 \qquad c_5^D = -\tfrac{1}{4} \qquad c_6 = -\tfrac{1}{2} \qquad c_7^N = -\tfrac{1}{2}.$$

□

We continue the proof of theorem 4 by determining all the remaining coefficients except d_3^D.

Lemma 9. *We have* $\tilde\beta_4(\phi, D, \rho) = -\frac{1}{2}\int_M\{\mathcal{G}_{1,ij}\phi_{;ij}+\mathcal{E}_1\phi+\mathcal{F}_{1,i}\phi_{;i}\}\rho\,\mathrm{d}x - \frac{1}{2}\int_{C_N}\mathcal{G}_{1,mm}(\mathcal{B}\phi)\rho\,\mathrm{d}y + \int_{C_D}\{\frac{1}{2}\mathcal{G}_{1,mm}\phi\rho_{;m} - \frac{1}{4}\mathcal{G}_{1,mm}L_{aa}\phi\rho + d_3^D\mathcal{G}_{1,mm;m}\phi\rho - \frac{5}{16}\mathcal{G}_{1,am}\phi_{:a}\rho - \frac{5}{16}\mathcal{F}_{1,m}\phi\rho\}\,\mathrm{d}y.$

Proof. Throughout the proof of this lemma, we shall let $M := S^1 \times [0, 1]$ with the flat metric and usual parameters (y, z). Let $\Delta_0 := -\partial_y^2 - \partial_z^2$ be the associated Laplacian.

Step 1. We use gauge invariance to determine the coefficients d_2, and d_5^D which appear in lemma 8. For $f \in C^\infty(M)$, let $\mathcal{D}_0 := \Delta_0 + f$ and let

$$D_t := \mathrm{e}^{tf}(\partial_t + \mathcal{D}_0)\mathrm{e}^{-tf} - \partial_t = \Delta_0 + 2tf_{;i}\nabla_i + tf_{;ii} - t^2 f_{;i}^2.$$

Here $\nabla_i = \partial_i$. We take pure Dirichlet boundary conditions so $C_N = \emptyset$. Let $u_0 := u_{\phi,\mathcal{D}}$ and let $u := \mathrm{e}^{tf}u_0$. We compute:

$$(\partial_t + D_t)u = \mathrm{e}^{tf}(\partial_t + \mathcal{D}_0)u_0 = 0 \qquad \mathcal{B}u = 0 \qquad \text{and} \qquad u|_{t=0} = u_0|_{t=0} = \phi.$$

Consequently $u = u_{\phi,D}$ so $\beta(\phi, D, \rho) = \int_M \mathrm{e}^{tf}u_0\rho$. Therefore,

$$\beta_4(\phi, D, \rho) = \beta_4(\phi, \mathcal{D}_0, \rho) + \beta_2(\phi, \mathcal{D}_0, f\rho) + \tfrac{1}{2}\beta_0(\phi, \mathcal{D}_0, f^2\rho).$$

We have $\Omega = 0$ and $E = -f$ for $\mathcal{D}_0$. We use theorem 3 to see that

$$\beta_0(\phi, \mathcal{D}_0, f^2\rho) = \int_M f^2\phi\rho\,\mathrm{d}x$$

$$\beta_2(\phi, \mathcal{D}_0, f\rho) = -\int_M f\rho(\Delta_0 + f)\phi - \int_{C_D}\phi(f\rho)_{;m}\,\mathrm{d}y$$

$$\beta_4(\phi, \mathcal{D}_0, \rho) = \tfrac{1}{2}\int_M\{(\Delta_0 + f)\phi\cdot(\Delta_0 + f)\rho\}\,\mathrm{d}x$$
$$+\int_{C_D}\{\tfrac{1}{2}((\Delta_0 + f)\phi)_{;m}\rho + \tfrac{1}{2}\phi((\Delta_0 + f)\rho)_{;m} - \tfrac{1}{8}f_{;m}\phi\rho\}\,\mathrm{d}y.$$

We have $D_0 = \Delta_0$. We use theorem 3 to compute $\beta_4(\phi, \Delta_0, \rho)$ and to see that:

$$\tilde\beta_4(\phi, D, \rho) = \tfrac{1}{2}\int_M f(\phi\Delta_0\rho - \rho\Delta_0\phi)\,\mathrm{d}x + \int_{C_D}\{\tfrac{1}{2}f\phi_{;m}\rho - \tfrac{1}{2}f\phi\rho_{;m} - \tfrac{1}{8}f_{;m}\rho\phi\}\,\mathrm{d}y.$$

We use Green's formula $\int_M(\alpha\Delta_0\beta-\beta\Delta_0\alpha)\,\mathrm{d}x=\int_{C_D}(\alpha\beta_{;m}-\beta\alpha_{;m})$ to see that

$$\int_M f(\phi\Delta_0\rho-\rho\Delta_0\phi)\,\mathrm{d}x=\int_M\{\rho(\Delta_0(f\phi)-f\Delta_0\phi)\}\,\mathrm{d}x+\int_{C_D}\{f\phi\rho_{;m}-\rho(f\phi)_{;m}\}\,\mathrm{d}y.$$

Consequently,

$$\tilde\beta_4(\phi,D,\rho)=\tfrac{1}{2}\int_M\{(-2f_{;i}\phi_{;i}-f_{;ii}\phi)\rho\}\,\mathrm{d}x-\tfrac{5}{8}\int_{C_D}f_{;m}\rho\phi\,\mathrm{d}y.$$

Since $\mathcal{F}_{1,i}=2f_{;i}$ and $\mathcal{E}_1=f_{;ii}$, we have

$$c_6=-\tfrac{1}{2}\qquad d_2=-\tfrac{1}{2}\qquad\text{and}\qquad d_5^D=-\tfrac{5}{16}.$$

Step 2. We take pure Neumann boundary conditions. Let

$$D:=\Delta_0+t(a\partial_z\partial_y+bz\partial_z^2+c\partial_z).$$

Let $\phi=\phi(y)$ depend only on the tangential variable and let $u_0=u_{\phi,\Delta_0}$. Because we are taking Neumann boundary conditions, u_0 only depends on the tangential variable so

$$(\partial_t+D)u_0=0\qquad\mathcal{B}u_0=0\qquad\text{and}\qquad u_0|_{t=0}=\phi.$$

Consequently, $u_{\phi,D}=u_0$. This implies that $\tilde\beta_4(\phi,D,\rho)=0$ so:

$$0=d_3^N=d_4^N=d_5^N.$$

Step 3. We take pure Dirichlet boundary conditions. Let

$$\mathcal{D}:=\mathrm{e}^{-\sqrt{-1}y}\Delta_0\mathrm{e}^{\sqrt{-1}y}=\Delta_0-2\sqrt{-1}\partial_y+1.$$

Let $u_0=u_{1,\mathcal{D}}$; this function is independent of the angular parameter y and only depends on the normal parameter z. We use equation (2) to compute $\omega_y=\sqrt{-1}$, $\omega_z=0$, $\Omega=0$, and $E=0$. (It is at this stage that we are forced to consider more general operators despite the fact that the primary interest is in the scalar Laplacian.) Let $\rho=\rho(y)$. We use theorem 3 to see

$$\begin{aligned}\beta_4(1,\mathcal{D},\mathrm{e}^{\sqrt{-1}y}\rho)&=\tfrac{1}{2}\int_M(\Delta_0-2\sqrt{-1}\partial_y+1)(\mathrm{e}^{\sqrt{-1}y}\rho)\\&=\tfrac{1}{2}\int_M\mathrm{e}^{\sqrt{-1}y}\rho=\beta_4(\mathrm{e}^{\sqrt{-1}y},\Delta_0,\rho).\end{aligned}$$

Let $u:=\mathrm{e}^{\sqrt{-1}y}u_0$ and let $D_t:=\Delta_0+t\partial_y\partial_z-\sqrt{-1}t\partial_z$. Then

$$(\partial_t+D_t)u=(\partial_t+\Delta_0)\mathrm{e}^{\sqrt{-1}y}u_0+t\partial_z(u_0)(\partial_y-\sqrt{-1})\mathrm{e}^{\sqrt{-1}y}=0.$$

Let $\phi:=\mathrm{e}^{\sqrt{-1}y}$. Then $u_{\phi,D}=u=\mathrm{e}^{\sqrt{-1}y}u_0$ so

$$\beta_4(\mathrm{e}^{\sqrt{-1}y},D,\rho)=\beta_4(1,\mathcal{D},\mathrm{e}^{\sqrt{-1}y}\rho).$$

We show that $d_4^D=d_5^D$ and complete the proof by computing:

$$\begin{aligned}0&=\beta_4(\mathrm{e}^{\sqrt{-1}y},D,\rho)-\beta_4(1,\mathcal{D},\mathrm{e}^{\sqrt{-1}y}\rho)\\&=\beta_4(\mathrm{e}^{\sqrt{-1}y},D,\rho)-\beta_4(\mathrm{e}^{\sqrt{-1}y},\Delta_0,\rho)\\&=\sqrt{-1}\int_{C_D}\mathrm{e}^{\sqrt{-1}y}(d_4^D-d_5^D)\rho(y)\,\mathrm{d}y.\end{aligned}$$

□

To complete the proof of theorem 4, it only remains to evaluate the coefficient d_3^D. This is a one-dimensional problem. Let $M := [0, 1]$. We make a change of variables on the manifold $M \times [0, \infty)$ to evaluate the coefficient of d_3^D that mixes up the space and time variables. Let

$$\tilde z := z + tz^2 \qquad \tilde t := t$$
$$d\tilde z = (1 + 2tz)\,\mathrm{d}z + z^2\,\mathrm{d}t \qquad \mathrm{d}\tilde t = \mathrm{d}t$$
$$\partial_{\tilde z} = (1 + 2tz)^{-1}\partial_z \qquad \partial_{\tilde t} = \partial_t - z^2(1 + 2tz)^{-1}\partial_z.$$

Let $\tilde D := -\partial_{\tilde z}^2 + \tilde z^2\partial_{\tilde z}$. Let

$$\begin{aligned} D := \partial_{\tilde t} + \tilde D - \partial_t &= -z^2(1 + 2tz)^{-1}\partial_z - (1 + 2tz)^{-2}\partial_z^2 + 2t(1 + 2tz)^{-3}\partial_z \\ &\quad +(z + tz^2)^2(1 + 2tz)^{-1}\partial_z = \Delta_0 + t\{4z\partial_z^2 + (2z^3 + 2)\partial_z\} + \mathrm{O}(t^2). \end{aligned}$$

Let $\phi = 1$, let $\tilde\rho$ be identically one near $\tilde z = 0$, let $\tilde\rho$ be identically zero near $\tilde z = 1$, and let $\rho(z; t) := \tilde\rho(z + tz^2)$. We impose Dirichlet boundary conditions; they are preserved by this coordinate transformation. Since ρ is zero away from the left-hand edge of the interval, the principal of not feeling the boundary shows we can neglect the right-hand edge. Thus

$$\tilde\rho(\tilde z)u_{\phi,\tilde D}(\tilde z; \tilde t) = \rho(z; t)u_{\phi,D}(z; t) + \mathcal{E}(z, t)$$

where the error $\mathcal{E}$ vanishes to infinite order in t as $t \downarrow 0$. Consequently, we have:

$$\begin{aligned} \beta(\phi, \tilde D, \tilde\rho)(t) &= \int_0^\infty u_{\phi,\tilde D}(\tilde z; \tilde t)\tilde\rho(\tilde z)\,\mathrm{d}\tilde z \\ &= \int_0^\infty u_{\phi,D}(z; t)\tilde\rho(z + tz^2)(1 + 2tz)\,\mathrm{d}z + \mathrm{O}(t^3) \\ &= \int_0^\infty u_{\phi,D}(z; t)\{\tilde\rho(z) + \tilde\rho'(z)tz^2 + \tfrac12\tilde\rho''(z)t^2z^4)\}\cdot(1 + 2tz)\,\mathrm{d}z + \mathrm{O}(t^3) \\ &= \beta(\phi, D, \tilde\rho)(t) + t\beta(\phi, D, \tilde\rho' z^2 + 2z\tilde\rho)(t) \\ &\quad +t^2\beta(\phi, D, \tfrac12\tilde\rho'' z^4 + 2\tilde\rho' z^3)(t) + \mathrm{O}(t^3). \end{aligned}$$

We expand both sides and compare the powers of t^2 to see that:

$$\beta_4(1, \tilde D, \tilde\rho) = \beta_4(1, D, \tilde\rho) + \beta_2(1, D, \tilde\rho' z^2 + 2z\tilde\rho) + \beta_0(1, D, \tfrac12\tilde\rho'' z^4 + 2\tilde\rho' z^3).$$

We have $D_0 = \Delta_0$, $\mathcal{G}_{mm} = 4z$, and $\mathcal{F}_m = 2z^3 + 2$. Recall that $\tilde\rho$ is identically one near zero. We use lemma 9 to compute $\beta_n(1, D, \cdot)$:

$$\beta_4(1, D, \tilde\rho) = \int_{C_D}(4d_3^D - \tfrac58)\phi\tilde\rho$$
$$\beta_2(1, D, \tilde\rho' z^2 + 2z\tilde\rho) = \int_{C_D}(-2)\phi\tilde\rho$$
$$\beta_0(1, D, \tfrac12\tilde\rho'' z^4 + 2\tilde\rho' z^3) = 0$$
$$\beta_4(1, \tilde D, \tilde\rho) = \int_{C_D}(4d_3^D - \tfrac58 - 2)\phi\tilde\rho.$$

The operator $\tilde D$ is autonomous. It is not self-adjoint so we must use some care in applying theorem 3. We use equation (2) to compute:

$$\tilde D = -\partial_{\tilde z}^2 + \tilde z^2\partial_{\tilde z} \qquad \tilde D^* = -\partial_{\tilde z}^2 - \tilde z^2\partial_{\tilde z} - 2\tilde z$$
$$\omega_m(\tilde D) = -\tfrac12\tilde z^2 \qquad \omega_m(\tilde D^*) = \tfrac12\tilde z^2$$
$$E(\tilde D) = \tilde z - \tfrac14\tilde z^4 \qquad E(\tilde D^*) = 2\tilde z - \tilde z - \tfrac14\tilde z^4.$$

Consequently, we compute that

$$\beta_4(\phi, \tilde{D}, \tilde{\rho}) = \int_{C_D} \{\tfrac{1}{2}(-2\tilde{z}\rho)_{;m}\phi + \tfrac{1}{8}(\tilde{z})_{;m}\tilde{\rho}\phi\}.$$

This yields the relation

$$-1 + \tfrac{1}{8} = 4d_3^D - \tfrac{5}{8} - 2 \qquad \text{so} \quad d_3^D = \tfrac{7}{16}.$$

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